LECTURE NO 25

Magnetostatics

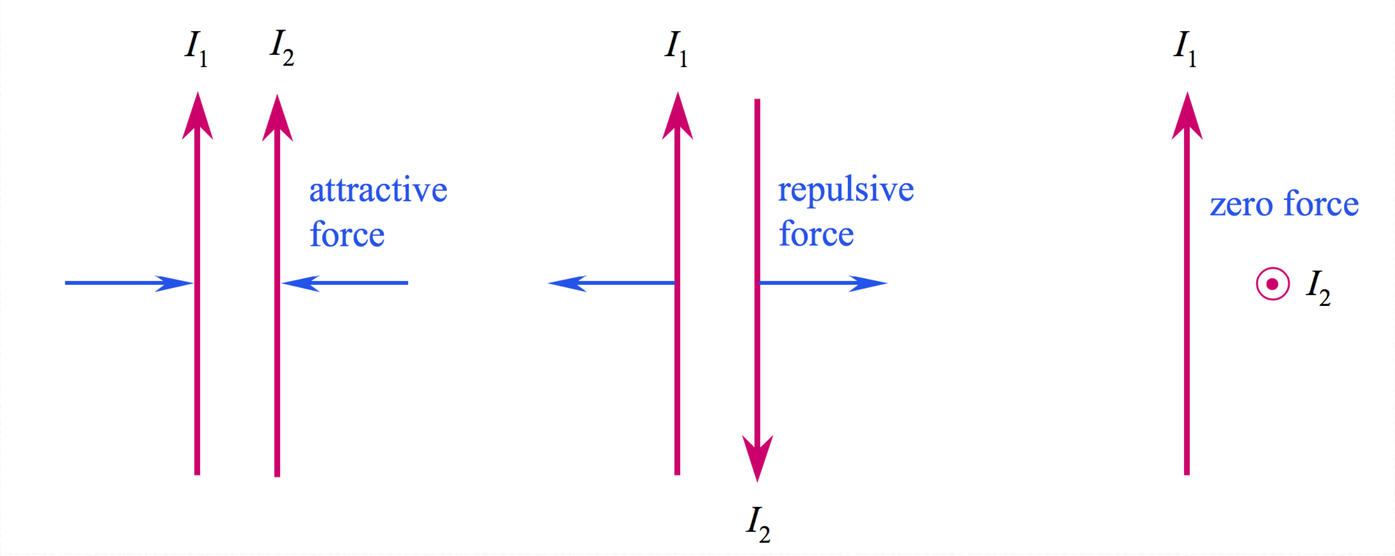
Topics

- Magneto-static fields,
- Biot-Savart's Law,
- Ampere's circuit law

Motivating the Magnetic Field Concept: Forces Between Currents

Magnetic forces arise whenever we have charges in motion. Forces between current-carrying wires present familiar examples that we can use to determine what a magnetic force field should look like:

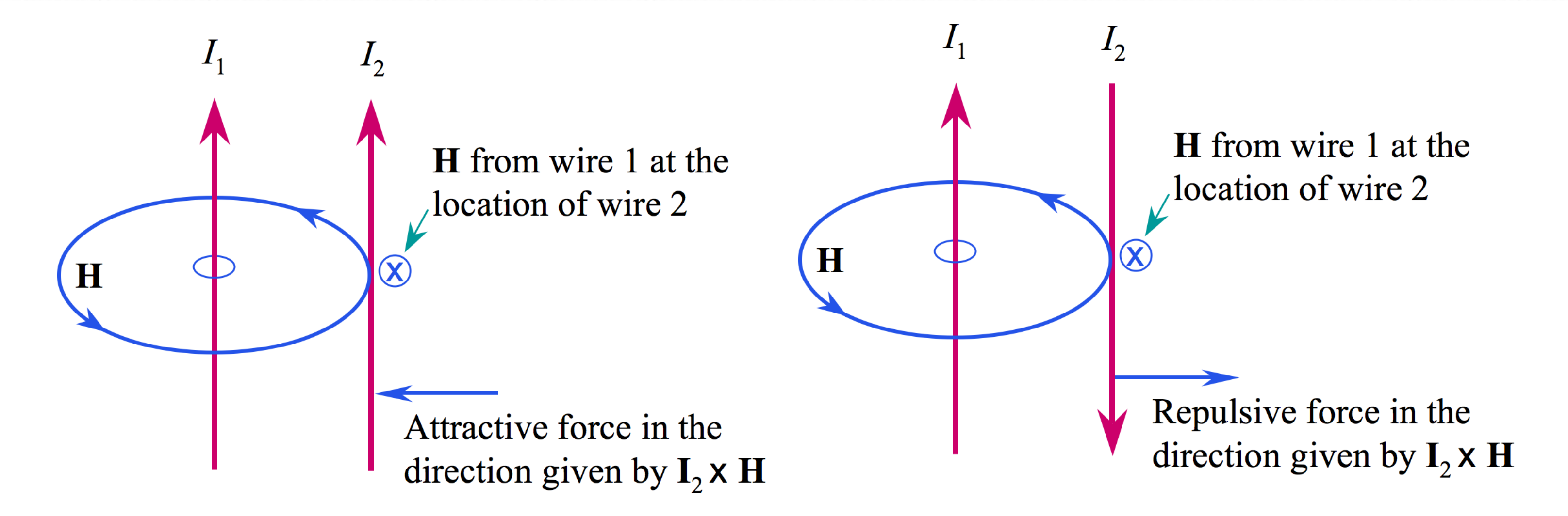
Here are the easily-observed facts:



How can we describe a force field around wire 1 that can be used to determine the force on wire 2?

Magnetic Field

The geometry of the magnetic field is set up to correctly model forces between currents that allow for any relative orientation. The magnetic field intensity, **H**, circulates *around* its source, I_1, in a direction most easily determined by the *right-hand rule*: Right thumb in the direction of the current, fingers curl in the direction of **H**



Note that in the third case (perpendicular currents), I_2 is in the same direction as **H**, so that their cross product (and the resulting force) is zero. The actual force computation involves a different field quantity, **B**, which is related to **H** through $\mathbf{B} = \mu_0\mathbf{H}$ in free space. This will be taken up in a later lecture. Our immediate concern is how to find **H** from any given current distribution.

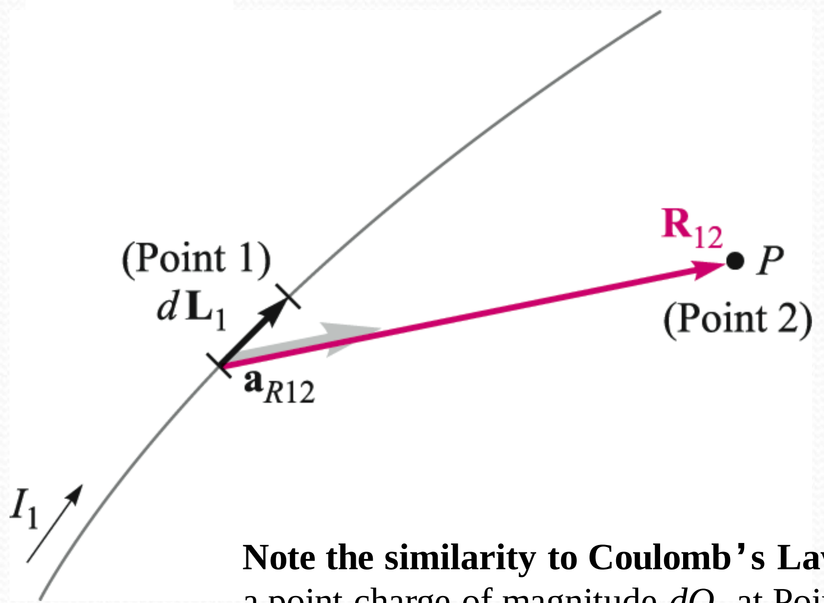
Biot-Savart Law

The Biot-Savart Law specifies the magnetic field intensity, **H**, arising from a "point source" current element of differential length *d***L.**

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

The units of **H** are [A/m]

Note in particular the inverse-square distance dependence, and the fact that the cross product will yield a field vector that points into the page. This is a formal statement of the right-hand rule



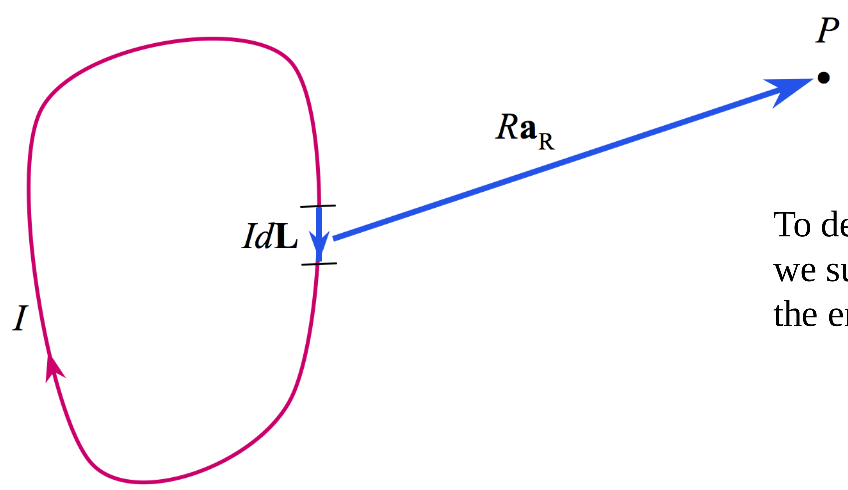
Note the similarity to Coulomb's Law, in which a point charge of magnitude dQ_1 at Point 1 would generate electric field at Point 2 given by:

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi\epsilon_0 R_{12}^2}$$

Magnetic Field Arising From a Circulating Current

At point *P*, the magnetic field associated with the differential current element *Id***L** is

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$



To determine the total field arising from the closed circuit path, we sum the contributions from the current elements that make up the entire loop, or

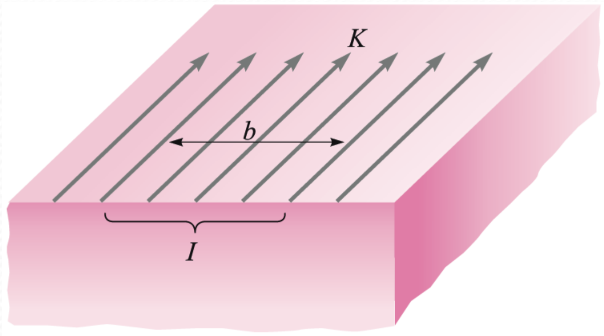
$$\boxed{\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

The contribution to the field at *P* from any portion of the current will be just the above integral evalated over just that portion.

Two- and Three-Dimensional Currents

On a surface that carries uniform surface current density **K** [A/m], the current within width *b* is

$$I = Kb$$



..and so the differential current quantity that appears in the Biot-Savart law becomes:

$$I\, d\mathbf{L} = \mathbf{K}\, dS$$

The magnetic field arising from a current sheet is thus found from the two-dimensional form of the Biot-Savart law:

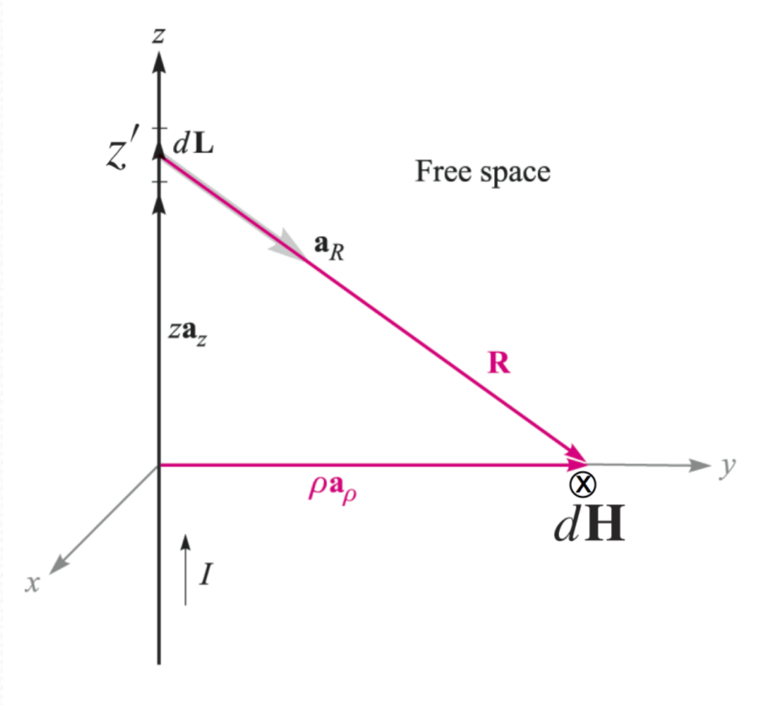
$$\boxed{\mathbf{H} = \int_S \frac{\mathbf{K} \times \mathbf{a}_R\, dS}{4\pi R^2}}$$

In a similar way, a **volume current** will be made up of three-dimensional current elements, and so the Biot-Savart law for this case becomes:

$$\boxed{\mathbf{H} = \int_{\mathrm{vol}} \frac{\mathbf{J} \times \mathbf{a}_R\, dv}{4\pi R^2}}$$

Example of the Biot-Savart Law

In this example, we evaluate the magnetic field intensity on the *y* axis (equivalently in the *xy* plane) arising from a filament current of infinite length in on the *z* axis.



Using the drawing, we identify:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho \mathbf{a}_\rho - z' \mathbf{a}_z$$

and so..
$$\mathbf{a}_R = \frac{\rho \mathbf{a}_\rho - z' \mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

so that:

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,dz'\mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

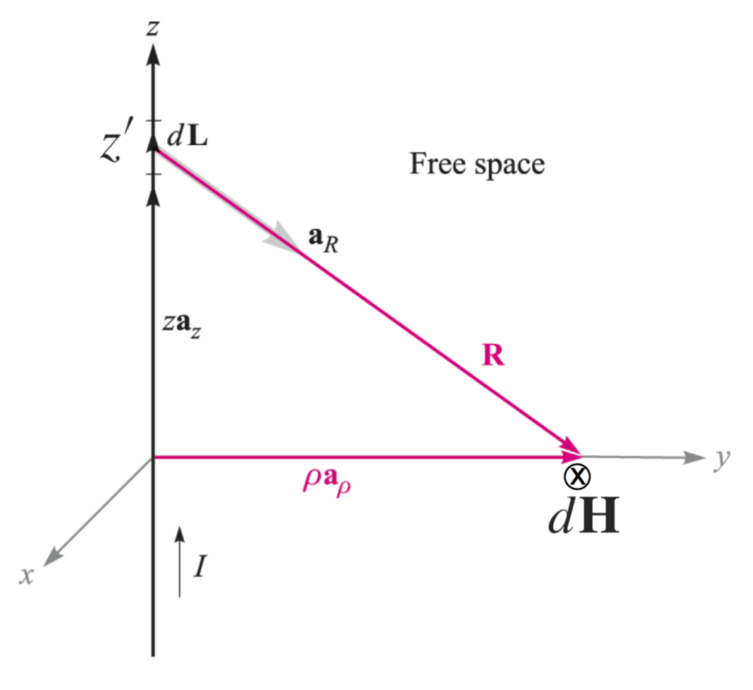
Example: continued

We now have: $$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Integrate this over the entire wire:

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\,dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$



..after carrying out the cross product

Example: concluded

Evaluating the integral:

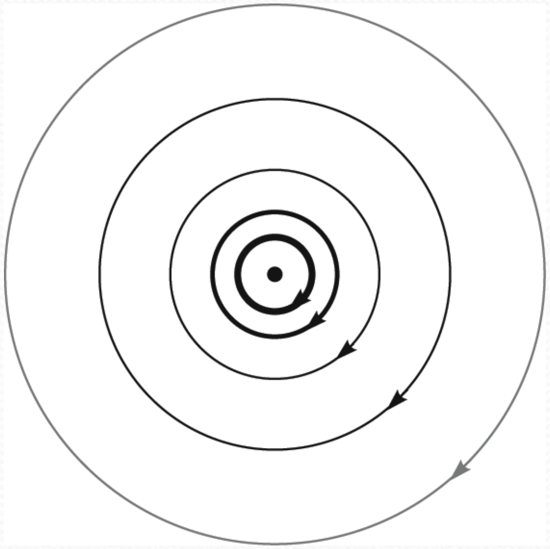
we have:

$$\mathbf{H} = \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho \, dz' \mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I \rho \mathbf{a}_\phi}{4\pi} \frac{z'}{\rho^2 \sqrt{\rho^2 + z'^2}} \Bigg|_{-\infty}^{\infty}$$

finally:

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi}$$



Current is into the page.
Magnetic field streamlines
are concentric circles, whose magnitudes
decrease as the inverse distance from the z axis

Field Arising from a Finite Current Segment

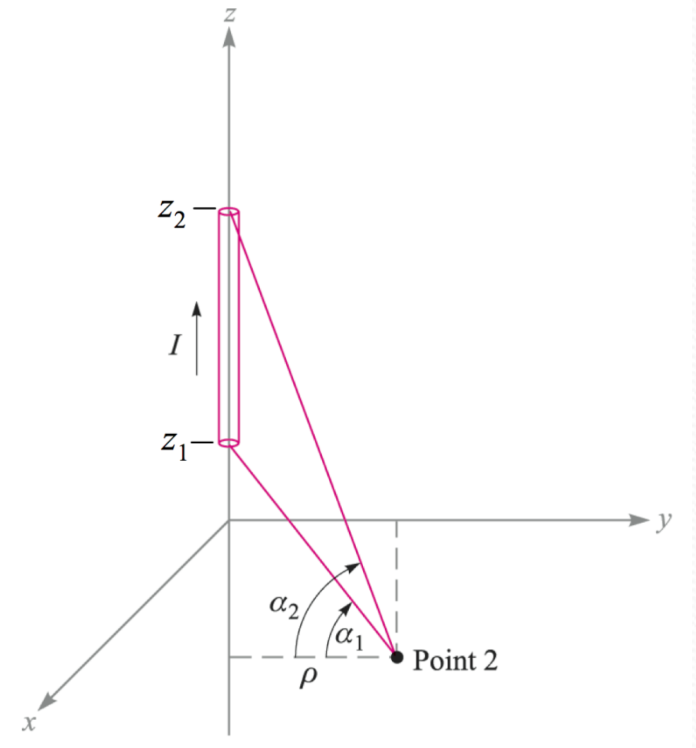
In this case, the field is to be found in the xy plane at Point 2.
The Biot-Savart integral is taken over the wire length:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$$= \int_{\rho \tan \alpha_1}^{\rho \tan \alpha_2} \frac{I dz \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z \mathbf{a}_z)}{4\pi(\rho^2 + z^2)^{3/2}}$$

..after a few additional steps (see Problem 7.8), we find:

$$\boxed{\mathbf{H} = \frac{I}{4\pi\rho}(\sin \alpha_2 - \sin \alpha_1)\mathbf{a}_\phi}$$



Another Example: Magnetic Field from a Current Loop

Consider a circular current loop of radius *a* in the *x*-*y* plane, which carries steady current *I*. We wish to find the magnetic field strength anywhere on the *z* axis.

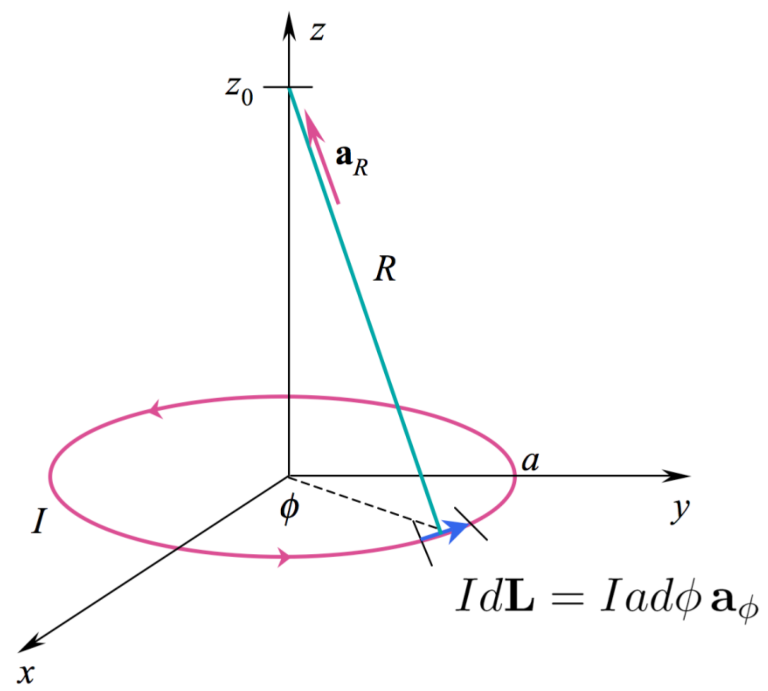
We will use the Biot-Savart Law:

$$\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where: $I d\mathbf{L} = I a d\phi \, \mathbf{a}_\phi$

$$R = \sqrt{a^2 + z_0^2}$$

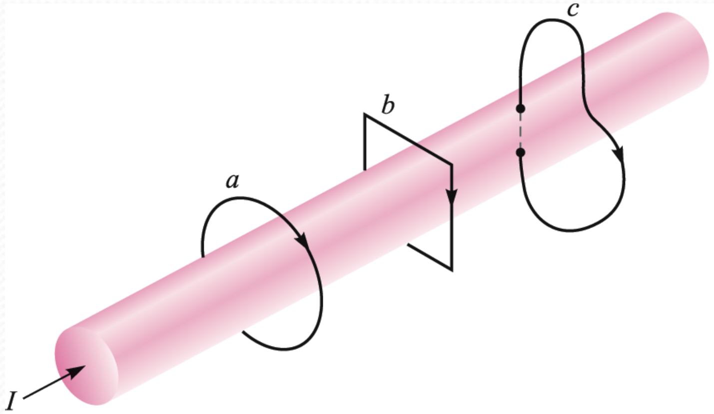
$$\mathbf{a}_R = \frac{z_0 \, \mathbf{a}_z - a \, \mathbf{a}_\rho}{\sqrt{a^2 + z_0^2}}$$



Ampere's Circuital Law

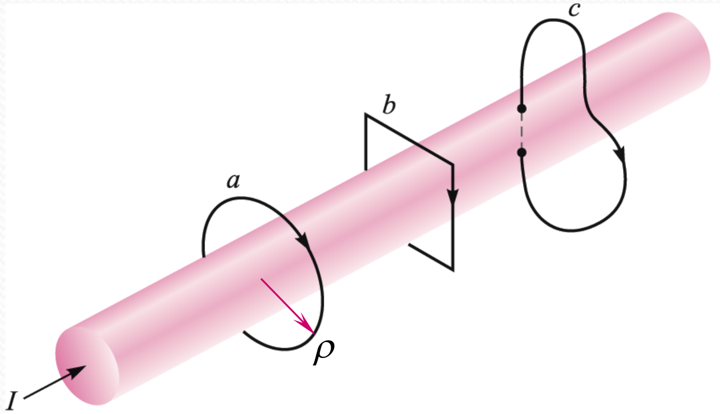
Ampere's Circuital Law states that the line integral of **H** about *any closed path* is exactly equal to the direct current enclosed by that path.

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = I}$$



In the figure at right, the integral of **H** about closed paths *a* and *b* gives the total current *I*, while the integral over path *c* gives only that portion of the current that lies within *c*

Ampere's Law Applied to a Long Wire



Symmetry suggests that **H** will be circular, constant-valued at constant radius, and centered on the current (z) axis.

Choosing path a, and integrating **H** around the circle of radius ρ gives the enclosed current, I:

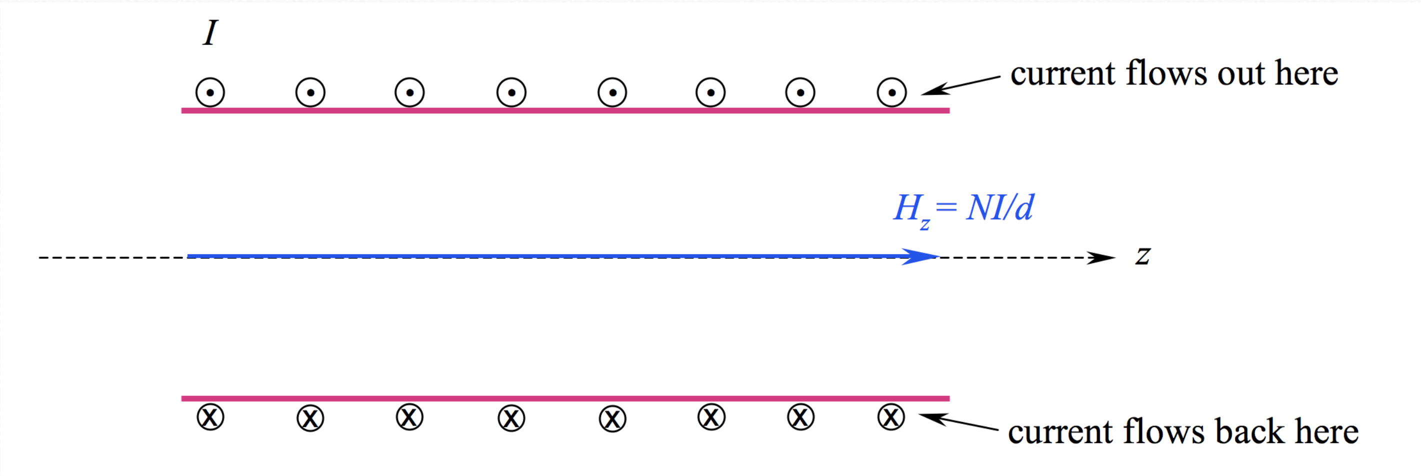
$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I$$

so that: $\boxed{H_\phi = \dfrac{I}{2\pi\rho}}$ as before.

Solenoid Field -- Off-Axis

To find the field within a solenoid, but off the *z* axis, we apply Ampere's Circuital Law in the following way:

The illustration below shows the solenoid cross-section, from a lengthwise cut through the *z* axis. Current in the windings flows in and out of the screen in the circular current path. Each turn carries current *I*. The magnetic field along the *z* axis is *NI/d* as we found earlier.

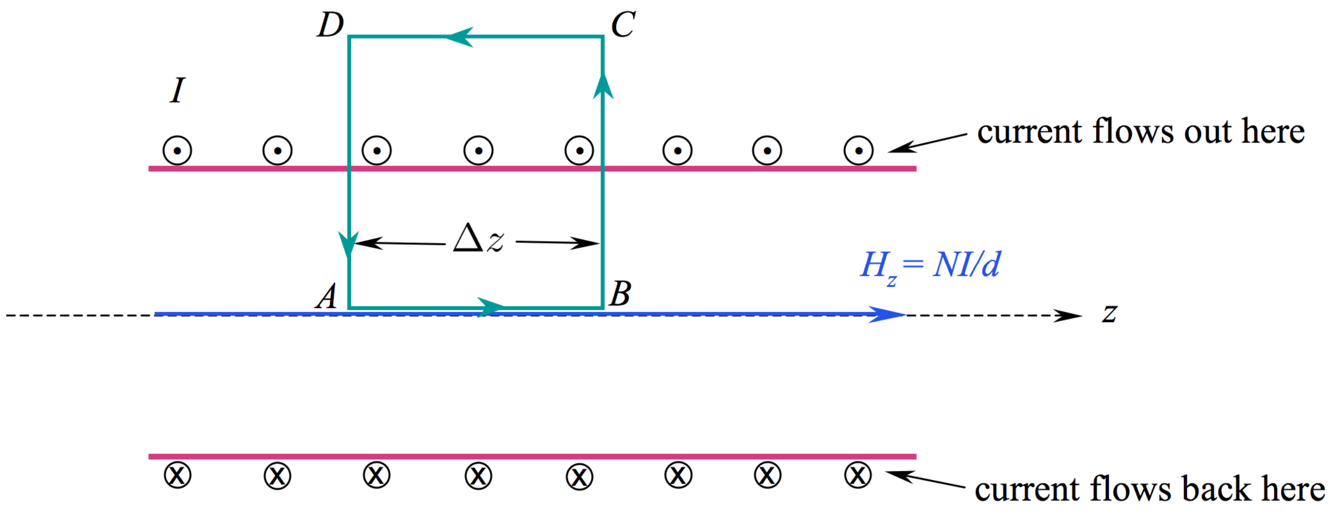


Application of Ampere's Law

Applying Ampere's Law to the rectangular path shown below leads to the following:

$$\oint \mathbf{H} \cdot d\mathbf{L} \;=\; \int_A^B H_z\, dz + \int_B^C H_\rho\, d\rho + \int_C^D H_{z,out}\, dz + \int_D^A H_\rho\, d\rho \;=\; I_{encl} \;=\; \frac{NI}{d}\Delta z$$

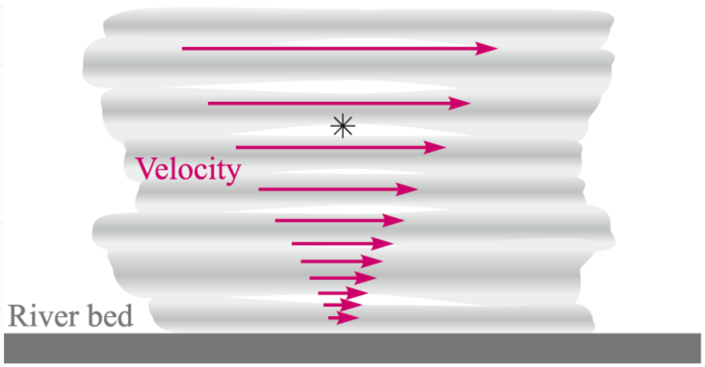
Where allowance is made for the existence of a radial *H* component, H_ρ



Visualization of Curl

Consider placing a small “paddle wheel” in a flowing stream of water, as shown below. The wheel axis points into the screen, and the water velocity decreases with increasing depth.

The wheel will rotate clockwise, and give a curl component that points into the screen (right-hand rule).



Positioning the wheel at all three orthogonal orientations will yield measurements of all three components of the curl. Note that the curl is directed normal to both the field and the direction of its variation.